Solutions to SMC KMC Workshop

Analysis by Vong Jun Yi and Lim Ho Hua

Section 1: Algebra

Student, 2017 #23
If $|x|+x+y=5$ and $x+|y|-y=10$, what is the value of $x+y$?
$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution 1: Trichotomy property of real numbers (Lim Ho Hua)
We can split into 4 cases:
Case 1: $x<0, y<0 \implies y=5, x-2y=10 \implies (x,y)=(20,5)$. This solution is rejected since it contradicts our inequality.
Case 2: $x>0, y<0 \implies 2x+y=5, x-2y=10 \implies (x,y)=(4,-3)$
Case 3: $x<0, y>0 \implies y=5, x=10 \implies (x,y)=(10,5)$. This solution is rejected since it contradicts our inequality.
Case 4: $x>0, y>0 \implies 2x+y=5, x=10 \implies (x,y)=(20,-15)$. This solution is rejected since it contradicts our inequality.
Therefore, $x+y=1 \implies \boxed{\textbf{(A)}}$.

Solution 2: Direct Substitution (Vong Jun Yi)

$$\begin{aligned} |x|+x+y&=5 \qquad \ldots\ldots(1) \\ x+|y|-y&=10 \qquad \ldots\ldots(2) \end{aligned}$$

From (1), we obtain

$$y=5-|x|-x \qquad \ldots\ldots(3)$$

Substitute (3) into (2):

$$\begin{aligned} x+|5-|x|-x|-5+|x|+x&=10 \\ |5-|x|-x|&=15-2x-|x| \\ 5-|x|-x&=\pm(15-2x-|x|) \end{aligned}$$

Hence, we have two cases. y is obtained by substituting corresponding values of x into (3)
Case 1:

$$5-|x|-x=15-2x-|x| \implies (x_1,y_1)=(10,-15)$$

Case 2:

$$5-|x|-x=2x+|x|-15 \implies |x|=\frac{20-3x}{2} \iff x=\pm\frac{20-3x}{2}$$

We shall split Case 2 into two further cases:

Case 2(a):

$$x = \frac{20 - 3x}{2} \implies (x_2, y_2) = (4, -3)$$

Case 2(b):

$$x = \frac{3x - 20}{2} \implies (x_3, y_3) = (20, -35)$$

By testing pairs (x, y) using (2), we see that only $(x_2, y_2) = (4, -3)$ works, therefore $x + y = 1 \implies \boxed{\textbf{(A)}}$.

Student, 2018 #21
How many real solutions does the equation $||4x - 3| - 2| = 1$ have?
$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution
We begin by removing the modulus step by step:

$$\begin{aligned} & ||4x - 3| - 2| = 1 \\ \iff & |4x - 3| - 2 = \pm 1 \end{aligned}$$

This gives us two cases, each of which will be divided further into two subcases.

Case 1: $|4x - 3| = 3 \implies 4x - 3 = \pm 3$
Case 1(a): $4x - 3 = 3 \iff x = 3/2$
Case 1(b): $4x - 3 = -3 \iff x = 0$

Case 2: $|4x - 3| = 1 \implies 4x - 3 = \pm 1$
Case 2(a): $4x - 3 = 1 \implies x = 1$
Case 2(b): $4x - 3 = -1 \implies x = 1/2$

Hence, we have 4 solutions in total $\implies \boxed{\textbf{(C)}}$.

Section 2: Combinatorics

Student, 2013 #20
(Student, 2013 #20) A box contains 900 cards numbered from 100 to 999. Any two cards have different numbers. Francois picks some cards and determines the sum of the digits on each of them. At least how many cards must he pick in order to be certain to have three cards with the same sum?
$\textbf{(A)}\ 51 \qquad \textbf{(B)}\ 52 \qquad \textbf{(C)}\ 53 \qquad \textbf{(D)}\ 54 \qquad \textbf{(E)}\ 55$

Solution

We have to consider the worst possible case, that is, we have to maximise the number of cards picked until three cards with the same digit sum is obtained.

From 100 to 999, there are 27 different digit sums obtainable $(1, 2, \ldots, 26, 27)$. To maximise number of cards Francois selects, we need to ensure that our cards have distinct digit sums, so Francois will be picking 27 cards for our first set of cards.

Notice that the digit sums 1 and 27 only appear once in the range 100 to 999, which we've already used for our first set. Francois is left with only 25 different sums/cards to pick for our second set of cards.

By Pigeonhole principle, the next card we pick will definitely share the same digit sum as two other cards. The answer is $27 + 25 + 1 = 53 \implies \boxed{\textbf{(C)}}$.

Student, 2015 #22

Blue and red rectangles are drawn on a blackboard. Exactly 7 of the rectangles are squares. There are 3 red rectangles more than blue squares. There are 2 red squares more than blue rectangles. How many blue rectangles are there on the blackboard?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 10$

Solution

Note: Squares are considered special cases of rectangles for this question.

We shall proceed by dividing splitting our rectangles as follows:

	Square	**Non-square**
Red	a	b
Blue	c	d

We are given:

$$\begin{aligned} a + c &= 7 & \ldots\ldots(1) \\ a + b &= c + 3 & \ldots\ldots(2) \\ a &= c + d + 2 & \ldots\ldots(3) \end{aligned}$$

By solving for b, c, d in terms of a, we have $b = 10 - 2a, c = 7 - a, d = 2a - 9$. After updating our table,

	Square	**Non-square**	**Total**
Red	a	$10 - 2a$	$10 - a$
Blue	$7 - a$	$2a - 9$	$a - 2$
Total	7	1	8

Note that $b+d=1$ means either $b=1$ or $d=1$, but $b=10-2a \neq 1$ as a would not be an integer. Therefore, $b=0, d=1 \implies a=5$.

The number of blue rectangles is $2+1=3 \implies \boxed{\textbf{(B)}}$.

Student, 2018 #23
There are $ 40%$ more girls than boys in a class. How many pupils are in this class if the probability that a two-person delegation selected at random consists of a girl and a boy is equal to 0.5? \
$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 38 \qquad \textbf{(E)}\ 42$

Let x be the number of boys and y the number of girls.

We know that $y=\frac{140}{100}x=\frac{7}{5}x$.

The number of ways to choose a two-person delegation that consists of a girl and a boy is xy.
The number of ways to choose a two-person delegation without restrictions is $\binom{x+y}{2}=\frac{1}{2}(x+y)(x+y-1)$.

Therefore, the probability of choosing a two-person delegation that consists of a girl and a boy is

$$\begin{aligned}\frac{xy}{\frac{1}{2}(x+y)(x+y-1)}&=\frac{1}{2}\\4xy&=(x+y)(x+y-1)\\4x\left(\frac{7}{5}x\right)&=\left(x+\frac{7}{5}x\right)\left(x+\frac{7}{5}x-1\right)\\\frac{28}{5}x^2&=\frac{144}{25}x^2-\frac{12}{5}x\\4x^2&=60x\\x(x-15)&=0\end{aligned}$$

Reject $x=0$. Thus, $x=15, y=21 \implies x+y=36 \implies \boxed{\textbf{(C)}}$.

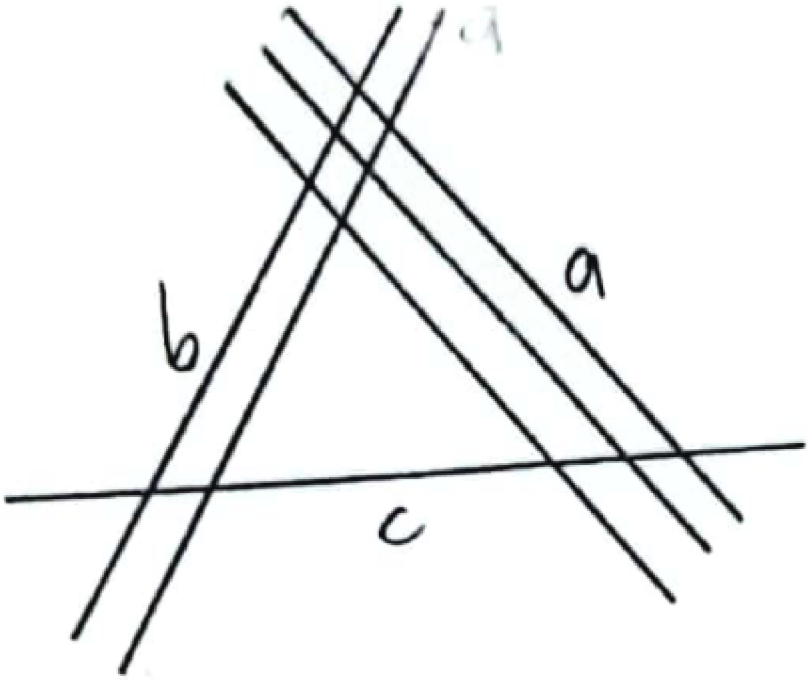
Section 3: Geometry

Student, 2013 #23
There are some straight lines drawn on the plane. Line a intersects exactly three other lines and line b intersects exactly four other lines. Line c intersects exactly n other lines, with $n \neq 3, 4$. Determine the number of lines drawn on the plane.
$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution

Observation: If we have k lines such that no two of them have the same gradient, all of the lines will intersect exactly $k-1$ other lines.

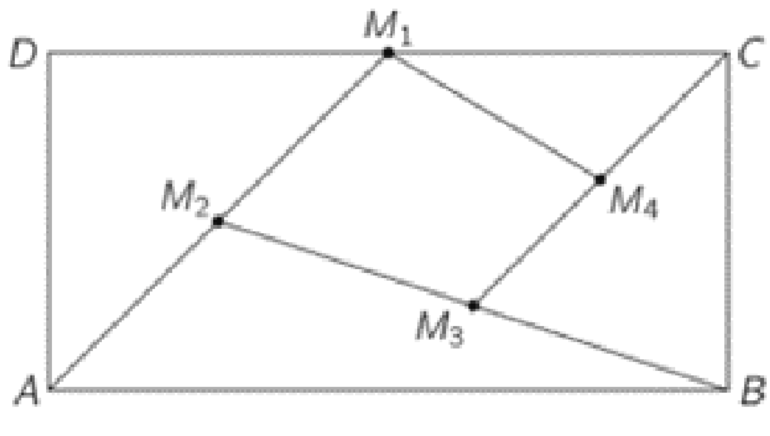
To ensure that each line intersects a different number of lines, we need to introduce parallel lines to our sketch.



Since line c intersects 5 other lines only, this arrangement is valid.
In total, there are 6 lines.
The answer is $\boxed{\textbf{(C)}}$.

Student, 2015 #21

In the rectangle $ABCD$ shown in the figure, M_1 is the midpoint of CD, M_2 is the midpoint of AM_1, M_3 is the midpoint of BM_2 and M_4 is the midpoint of CM_3. Find the ratio between the areas of the quadrilateral $M_1M_2M_3M_4$ and of the rectangle $ABCD$.



(A) $\frac{7}{16}$ **(B)** $\frac{3}{16}$ **(C)** $\frac{7}{32}$ **(D)** $\frac{9}{32}$ **(E)** $\frac{1}{5}$

Solution 1: Coordinate bashing (Vong Jun Yi)

We can mark points A, B, C, D with coordinates $(0,0),(x,0),(x,y),(0,y)$ respectively.

Note that the coordinates of a midpoint of a line segment is given by

$$(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).$$

Since M_1 is the midpoint of $\overline{CD}$, M_1 is located at $\left(\frac{1}{2}x,y\right)$.
Since M_2 is the midpoint of $\overline{AM_1}$, M_2 is located at $\left(\frac{1}{4}x,\frac{1}{2}y\right)$.
Since M_3 is the midpoint of $\overline{BM_2}$, M_3 is located at $\left(\frac{5}{8}x,\frac{1}{4}y\right)$.
Since M_4 is the midpoint of $\overline{CM_3}$, M_4 is located at $\left(\frac{13}{16}x,\frac{5}{8}y\right)$.
Since no particular rectangle is specified, we can set $(x,y)=(1,1)$.

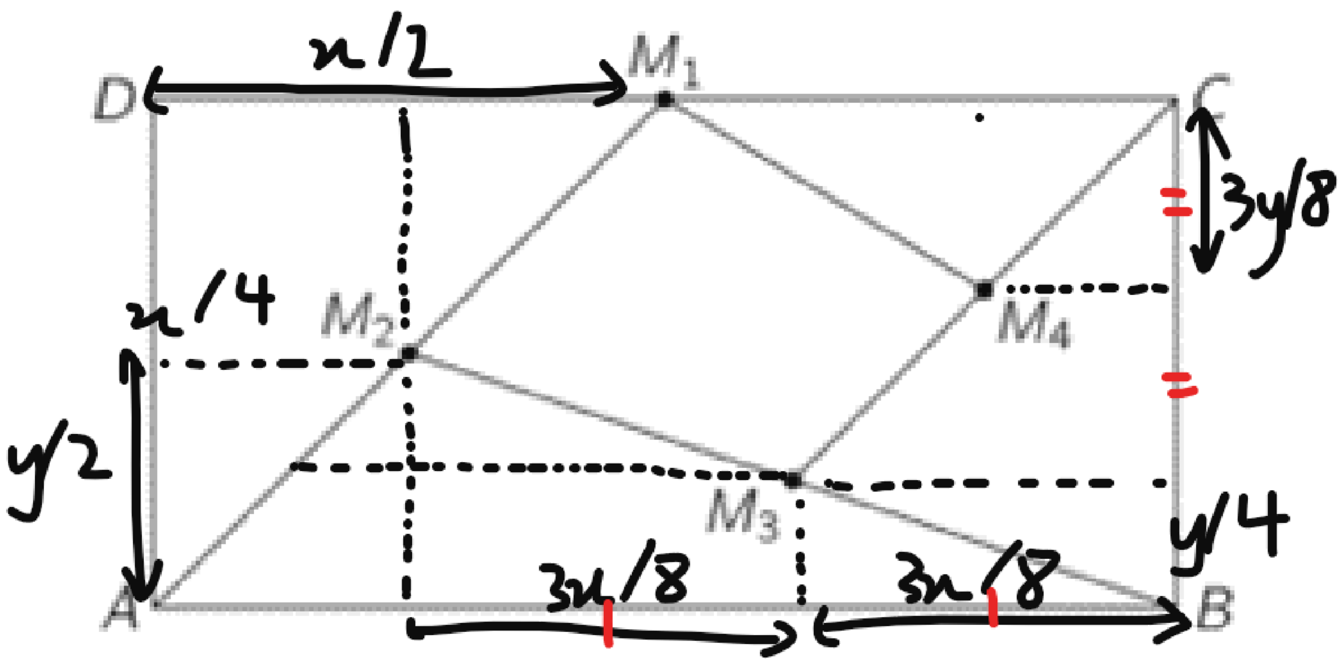
Using shoelace theorem, we have

$$[M_1M_2M_3M_4]=\frac{1}{2}\begin{vmatrix}1/2 & 1/4 & 5/8 & 13/16 & 1/2\\ 1 & 1/2 & 1/4 & 5/8 & 1\end{vmatrix}$$

$\therefore [M_1M_2M_3M_4]=\frac{7}{32}$
The answer is $\boxed{\textbf{(C)}}$.

Solution 2: Triangle areas (Lim Ho Hua)

Let the base and height of the rectangle be x and y. We shall mark the heights and bases of our triangles $\triangle AM_1D, \triangle AM_2B, \triangle BM_3C, \triangle M_1M_4C$.



$$[\triangle AM_1D] = \frac{1}{2}\left(\frac{1}{2}x\right)y = \frac{1}{4}xy$$

$$[\triangle AM_2B] = \frac{1}{2}x\left(\frac{1}{2}y\right) = \frac{1}{4}xy$$

$$[\triangle BM_3C] = \frac{1}{2}\left(\frac{3}{8}x\right)y = \frac{3}{16}xy$$

$$[\triangle M_1M_4C] = \frac{1}{2}\left(\frac{1}{2}x\right)\left(\frac{3}{8}y\right) = \frac{3}{32}xy$$

$$[M_1M_2M_3M_4] = xy - \frac{1}{4}xy - \frac{1}{4}xy - \frac{3}{16}xy - \frac{3}{32}xy = \frac{7}{32}xy$$

$$\therefore \frac{[M_1M_2M_3M_4]}{ABCD} = \frac{7}{32}$$

The answer is $\boxed{\textbf{(C)}}$.

Student, 2016 #22

A cube is dissected into 6 pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 2, 5, 10, 11 and 14. What is the volume of the sixth pyramid?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 9 \qquad \textbf{(E)}\ 12$

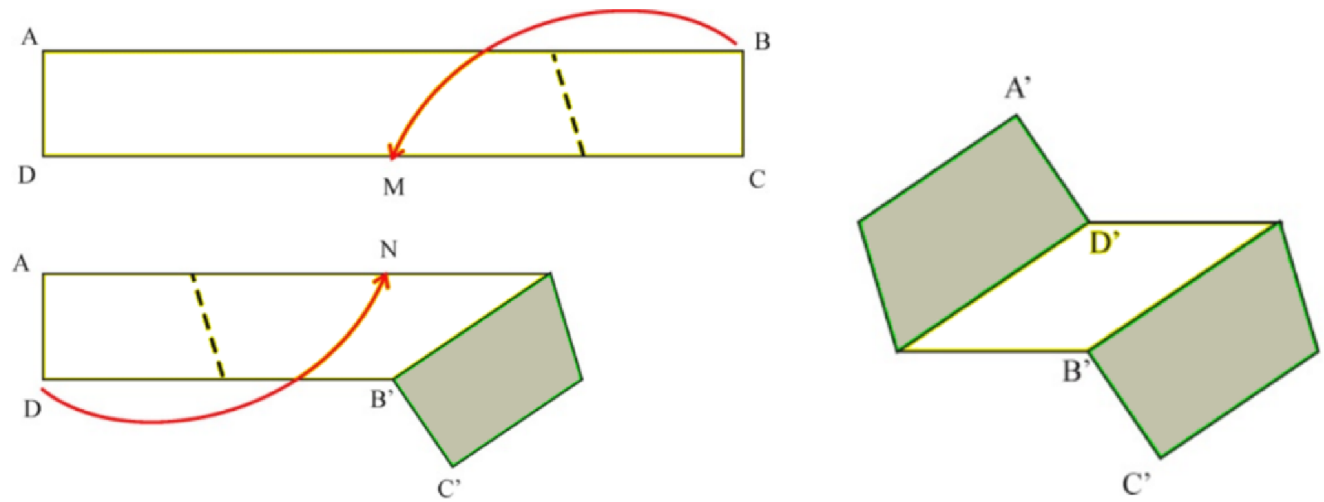
Solution

Recall $V_{\text{pyramid}} = \frac{1}{3}bh$, where b is the base area and h is the height of the pyramid.
We denote the height of the ith pyramid as h_i

Notice that $h_1 + h_2 = h_3 + h_4 = h_5 + h_6$.
By multiplying throughout by $\frac{1}{3}b$, we obtain
$\frac{1}{3}bh_1 + \frac{1}{3}bh_2 = \frac{1}{3}bh_3 + \frac{1}{3}bh_4 = \frac{1}{3}bh_5 + \frac{1}{3}bh_6 \implies V_1 + V_2 = V_3 + V_4 = V_5 + V_6$.
By finding two pairs of volumes with the same sum, we can deduce the third sum.
$16 = 5 + 11 = 2 + 14 = 10 + V_6 \implies V_6 = 6 \implies \boxed{\textbf{(C)}}$.

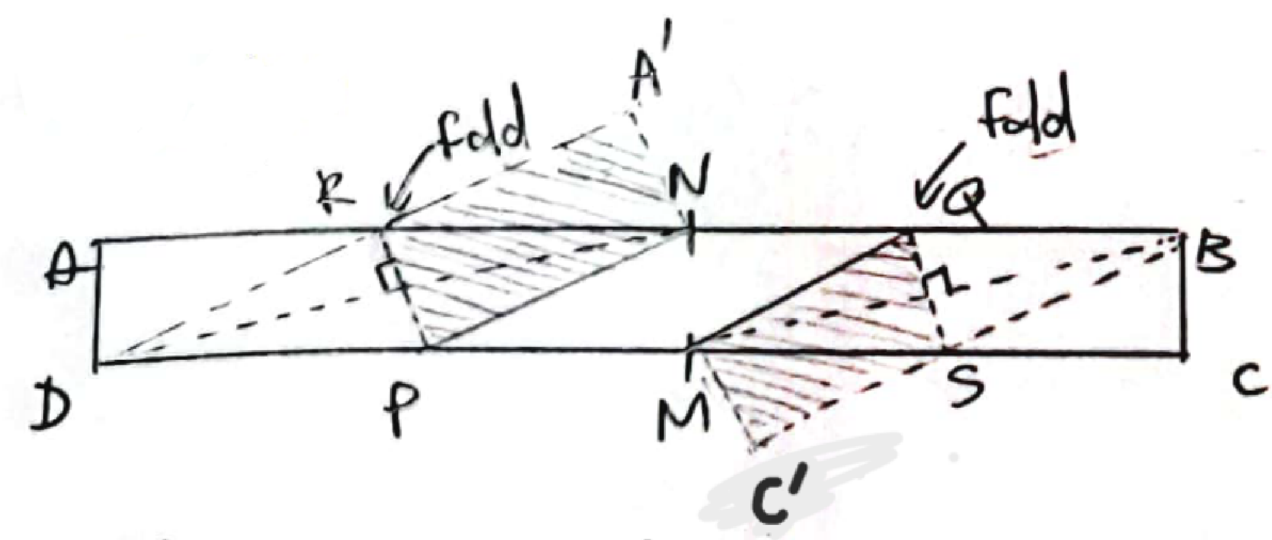
Student, 2016 #23

A rectangular strip ABCD of paper 5 cm wide and 50 cm long is white on one side and grey on the other. Folding the strip, Cristina makes the vertex B coincide with the midpoint M of the side CD. Folding again, she makes the vertex D coincide with the midpoint N of the side AB. What is the area (in cm^2) of the visible white part of the strip in the last picture?



$\textbf{(A)}\ 50 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 62.5 \qquad \textbf{(D)}\ 100 \qquad \textbf{(E)}\ 12$

Solution



NOTE: We cannot assume the length of the fold is equal to the width of the paper strip since $PR \neq A'N = AD$ and $QS \neq C'M = BC$.

We know that $\tan\angle NDM = \dfrac{5}{25} \iff \angle NDP = \arctan\dfrac{1}{5}$.

$DP = PN \implies \angle NDP = \angle DNP \implies \angle NPM = 2\arctan\dfrac{1}{5}$.

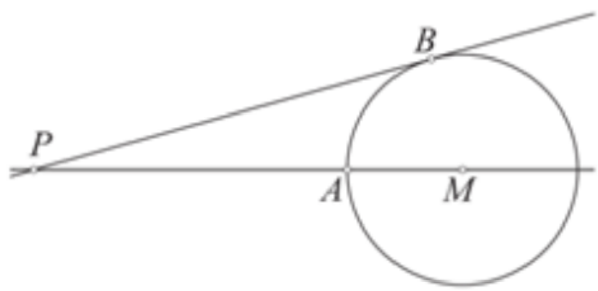
Similarly, we obtain $\angle MQN = 2\arctan\dfrac{1}{5}$, so $PN||QM$, then $PNQM$ is a parallelogram.

$$\begin{aligned}\frac{MN}{PM} &= \tan\left(2\arctan\frac{1}{5}\right)\\ \frac{5}{PM} &= \frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\\ PM &= 12\end{aligned}$$

Thus, the area of the parallelogram $PNQM$ is $12 \times 5 = 60 \implies \boxed{\textbf{(B)}}$.

Junior, 2017 #24

Points A and B are on the circle with centre M. PB is tangent to the circle at B. The distances PA and MB are integers. We know that $PB = PA + 6$. How many possible values are there for the length of MB?



$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Solution 1: Pythagorean Theorem (Lim Ho Hua)

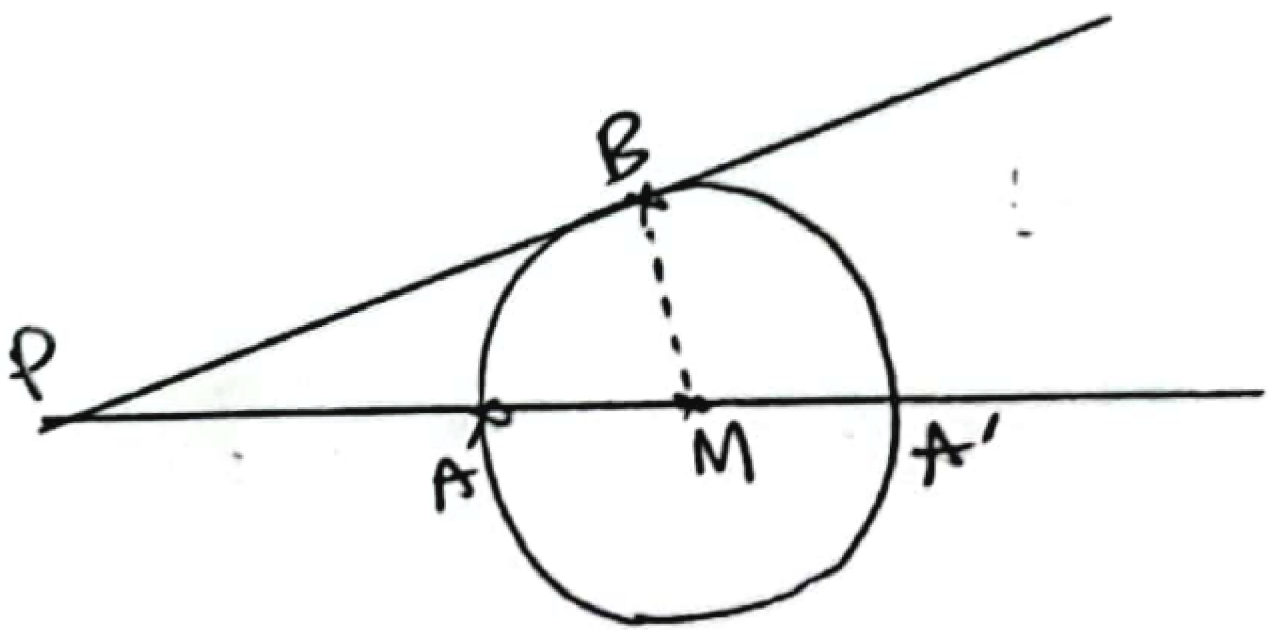
$$\begin{aligned}PM^2 &= PB^2 + BM^2\\ (PA+AM)^2 &= (PA+6)^2 + BM^2\end{aligned}$$

Let $r = AM = BM$.

$$\begin{aligned}PM^2 &= PB^2 + BM^2\\ (PA+r)^2 &= (PA+6)^2 + r^2\\ PA^2 + 2r(PA) + r^2 &= PA^2 + 12PA + 36 + r^2\\ PA(r-6) &= 18\end{aligned}$$

Since we need PA to be an integer, PA must be a positive divisor of 18. There are 6 positive divisors of 18, so the answer is $\boxed{\textbf{(D)}}$.

Solution 2: Power of Point (Vong Jun Yi)
Extend AM so that it meets the circle at another point A'.



By Power of a Point Theorem, $PB^2 = PA \cdot PA'$
Then,

$$\begin{aligned}(PA+6)^2 &= PA(PA+2r)\\ PA^2+12PA+36 &= PA^2+2r(PA)\\ PA(r-6) &= 18\end{aligned}$$

The conclusion is the same as in Solution 1. The answer is $\boxed{\textbf{(D)}}$.

Section 4: Number Theory

Student, 2014 #21
Tom wants to write several distinct positive integers, none of them exceeding 100. Their product should not be divisible by 54. At most how many integers can he write?
$\textbf{(A)}\ 8 \qquad \textbf{(B)}\ 17 \qquad \textbf{(C)}\ 68 \qquad \textbf{(D)}\ 69 \qquad \textbf{(E)}\ 90$

Solution
Note that the prime factorisation of 54 is $2 \cdot 3^3$.

We shall consider a few optimal cases.
Case 1: Our product is divisible by at most 3^3. This means all of Tom's integers are odd; at most 50 integers can be written.

Case 2: Our product is divisible by at most $2 \cdot 3^2$. The product of all multiples of 3 that Tom has selected is divisible by at most 3^2. For example, we can select 3 and 6 as our multiples, not 3 and 9

since their product will be divisible by 3^3. At most $100 - 33 + 2 = 69$ can be written.

Case 2 is better than Case 1. The answer is $\boxed{\textbf{(D)}}$.

Student, 2015 #24

In the word KANGAROO, Bill and Bob replace the letters by digits, so that the resulting numbers are multiples of 11. They each replace different letters by different digits and the same letters by the same digits ($K \neq 0$). Bill obtains the largest possible such number and Bob the smallest. In both cases one of the letters is replaced by the same digit. Which digit is this?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

Since $\overline{KANGAROO}$ is divisible by 11, we can invoke divisibility by 11 rule (alternating sum of digits need to be equal to $0 \pmod{11}$).

Hence, we have

$$K - A + N - G + A - R + O - O \equiv 0 \pmod{11}$$
$$K + N \equiv G + R \pmod{11}$$

Observe that the divisibility of $\overline{KANGAROO}$ by 11 is independent of A and O.
In Bill's case, we begin by letting $K = 9, A = 8, N = 7$. Then, $G + R \equiv 16 \equiv 5 \pmod{11}$.
The next largest number we can pick for G is 6 but no such R allows this.
Then, we pick $G = 5$ instead, which gives $R = 0$.
Therefore, Bill obtained 98758066.

In Bob's case, we begin by letting $K = 1, A = 0, N = 2$. Then, $G + R \equiv 3 \pmod{11}$.
The next smallest value we can pick for G is 3, so $R = 0$ but $A \neq R$.
Then, we pick $G = 4$ instead but no such R allows this.
Now we pick $G = 5$, which gives $R = 9$ as a possible solution.
Therefore, Bill obtained 10250933.

In both numbers, $G = 5$, so our answer is $\boxed{\textbf{(D)}}$.

Student, 2017 #24

How many three-digit positive integers ABC exist, such that $(A + B)^C$ is a three-digit integer and an integer power of 2?
Note: An integer power of 2 is a number in the form 2^k , where k is an integer.

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 16 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 21$

Solution

We shall begin by considering what the possible values of $(A + B)$ are.

When $A+B=2$, $7 \leq C \leq 9$.
When $A+B=4$, $C=4$
When $A+B=8$, $C=3$.
When $A+B=16$, $C=2$.
No values of C exist for $A+B>16$ such that $(A+B)^C$ is a three-digit integer. We also make sure to dismiss $A+B<0$ since A and B are always positive.

There exists 2 pairs (A,B) for $A+B=2$.
There exists 4 pairs (A,B) for $A+B=4$.
There exists 8 pairs (A,B) for $A+B=8$.
There exists 3 pairs (A,B) for $A+B=16$.

In total, there are $2 \times 3 + 4 + 8 + 3 = 21$ possible triples (A,B,C). The answer is $\boxed{\textbf{(E)}}$.

Student, 2018 #24
Archimedes calculated 15!. The result is written on the board. Unfortunately two of the figures, the second and the tenth, are not visible. Which are these two figures?

$\textbf{(A)}\ 2\ \&\ 0 \qquad \textbf{(B)}\ 4\ \&\ 8 \qquad \textbf{(C)}\ 7\ \&\ 4 \qquad \textbf{(D)}\ 9\ \&\ 2 \qquad \textbf{(E)}\ 3\ \&\ 8$

Solution 1 (Lim Ho Hua)
We want to find x and y where $15! = \overline{1x0767436y000}$

Since $11 \mid 15!$,
$1-x+0-7+6-7+4-3+6-y+0-0+0 \equiv 0 \pmod{11} \iff x+y \equiv 0 \pmod{11}$

Since $9 \mid 15!$, $1+x+0+7+6+7+4+3+6+y+0+0+0 \equiv 0 \pmod{9} \iff x+y \equiv 2 \pmod{9}$

Hence, $x+y = 11k = 9m+2$, where k, m are nonnegative integers. Since $x+y$ is at most 18, $(k,m)=(1,1)$ yields $x+y=11$.

$\textbf{(A)}$ and $\textbf{(B)}$ are automatically eliminated.

Notice that $3000 = 4 \times 5 \times 10 \times 15$

We can find y by finding the last digit of $\dfrac{15!}{1000}$

Now $\dfrac{15!}{1000} = \dfrac{15!}{3000} \times 3 = 1 \times 2 \times 3 \times 6 \times 7 \times 8 \times 9 \times 11 \times 12 \times 13 \times 14 \times 3 \equiv 8 \pmod{10}$
Therefore, the answer is $\boxed{\textbf{(E)}}$.

Solution 2: Divisibility rule for 2^n (Vong Jun Yi)

Same as above after $\textbf{(A)}$ and $\textbf{(B)}$ are eliminated.

We shall find the highest power of 2 that divides $15!$, which we denote by p.

$$p = \left\lfloor \frac{15}{2} \right\rfloor + \left\lfloor \frac{15}{2^2} \right\rfloor + \left\lfloor \frac{15}{2^3} \right\rfloor = 11$$

This means 2^{11} divides $15!$. Recall that a number is divisible by 2^n if its last n digits is also divisible by 2^n.

From the first 3 trailing zeros, we can tell $15!$ is divisible by 8.

We require $\overline{y000}$ to be divisible by 16. In fact, $y = 2, 4, 8$ satisfies this criterion.

We require $\overline{6y000}$ to be divisible by 32 or $\overline{6y}$ to be divisible by 4. Only $y = 4, 8$ satisfies this criterion, so eliminate $\textbf{(C)}$.

We require $\overline{36y000}$ to be divisible by 64 or $\overline{36y}$ to be divisible by 8. Only $y = 8$ satisfies this criterion, so eliminate $\textbf{(D)}$.

The only valid answer is $\boxed{\textbf{(E)}}$. To confirm this, tests for divisibility by 7 and 13 can be carried out.